# "Screening" of universal van der Waals – Casimir terms by Coulomb gases in a fully-finite two-dimensional geometry

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**Abstract.** This paper is a continuation of a previous one [Jancovici B and Šamaj L, 2004 J. Stat. Mech. P08006] dealing with classical Casimir phenomena in semi-infinite wall geometries. In that paper, using microscopic Coulomb systems, the long-ranged Casimir force due to thermal fluctuations in conducting walls was shown to be screened by the presence of an electrolyte between the walls into some residual short-ranged force. Here, we aim to extend the study of the screening (cancellation) phenomena to universal Casimir terms appearing in the large-size expansions of the grand potentials for microscopic Coulomb systems confined in fully-finite 2D geometries, in particular the disc geometry. Two cases are solved exactly: the high-temperature (Debye-Hückel) limit and the Thirring free-fermion point. Similarities and fundamental differences between fully-finite and semi-finite geometries are pointed out.

## 1. Introduction

At zero temperature, fluctuations of the quantum electromagnetic field in vacuum manifest themselves via an attraction of two parallel ideal-conductor plates. This Casimir effect (for an introduction see [1]) has a universal character in the sense that it does not depend on the material constitution of the metallic plates. Casimir's result was extended to arbitrary temperatures and general dielectric plates [2, 3], and to ideal-conductor walls of arbitrary smooth shapes [4]. For a recent book and review see [5, 6]. The studied models can be divided according to the geometry of fluctuating walls into two basic sets: the semi-infinite systems, in which at least one of the spatial coordinates is unconstrained by the walls (e.g. two parallel plates), and the fully-finite systems (e.g. a sphere). The applied methods and observed Casimir phenomena usually depend on this classification.

As concerns semi-infinite geometries, in the high-temperature limit defined by the validity of the equipartitioning energy law, the Casimir force becomes purely entropic [7]; this force is usually called classical since it does not depend on Planck's constant. In the purely electrostatic models which do not incorporate the magnetic part of the Lorentz force due to charge currents, like the system of scalar photons [8], the Casimir force is divided by a factor 2.

As concerns fully-finite three-dimensional (3D) conductor systems [4, 9], in the high-temperature limit, the Casimir free energy does depend on Planck's constant. Furthermore, the presence of both electric and magnetic degrees of freedom is necessary for obtaining a Casimir effect. This is no longer true for classical 2D Coulomb fluids defined in fully-finite domains. There, the consideration of the pure Coulomb potential, defined as the solution of the 2D Poisson equation, leads to a universal Casimir term analogous to the one appearing in finite-size expansions of thermodynamic quantities for 2D critical systems with *short-range* interactions among constituents. To be more precise, it is known that, according to the principle of conformal invariance, for a finite system of characteristic size $R$, at a critical point, the (dimensionless) free energy has a large-$R$ expansion of the form [10, 11, 12, 13]

$$\beta F = AR^2 + BR - \frac{c\chi}{6}\ln R + \text{const} + \cdots, \tag{1.1}$$

where $\beta$ denotes the inverse temperature. The coefficients $A$ and $B$ of the bulk and surface parts are non-universal. The coefficient of the logarithmic Casimir term is universal, dependent only on the conformal anomaly number $c$ of the critical theory and on the Euler number $\chi$ of the manifold on which the system is confined. In general, $\chi = 2 - 2h - b$, where $h$ is the number of handles and $b$ the number of boundaries of the manifold ($\chi = 2$ for the surface of a sphere, $\chi = 1$ for a disc, $\chi = 0$ for an annulus or a torus). A simple example is the Gaussian model [14] which is critical at all temperatures, with the conformal anomaly number $c = 1$. At any temperature of the conducting regime, the grand potential of a classical 2D Coulomb system of characteristic size $R$ is supposed to exhibit a large-$R$ expansion of type (1.1), with however a + sign in front

of the logarithmic term:

$$\beta\Omega_{\text{Coulomb}} = AR^2 + BR + \frac{\chi}{6}\ln R + \text{const} + \cdots. \tag{1.2}$$

Plausible arguments for a critical-like behaviour were first given for Coulomb gases with periodic boundary conditions [15], then for Coulomb systems confined to a domain by (vacuum) plain hard walls [16], by inert ideal-conductor walls (i.e. when the electric potential obeys Dirichlet boundary conditions) [17] and finally by ideal-dielectric boundaries (i.e. when the electric potential obeys Neumann boundary conditions) [18]. The explicit checks were done in the Debye-Hückel limit [19], at the free-fermion point of the Thirring representation of the symmetric two-component plasma, based on the formalism developed in [20], and for the one-component plasma at $\beta = 2$ in a disc [16, 17]. Only recently, a direct derivation of the universal finite-size correction term was done, in the whole stability range of temperatures, for the specific cases of the symmetric two-component plasma living on the surface of a sphere [21] and in a disc surrounded by vacuum [22]. In both cases, the universal prefactor to the $\ln R$ correction term in (1.2) was related to the bulk second moment of the density structure factor which is known [23]. This is very different from the semi-infinite geometries where the universality of the Casimir force is related to the second moment sum rule for the *charge* structure factor [17, 24].

It is generally believed that, for semi-infinite systems, the presence of an electrolyte between fluctuating conductor walls screens the long-ranged Casimir force to some residual short-ranged force [25, 26]. The study of Casimir phenomena via fully microscopic Coulomb models has the advantage of a coherent description of electrostatic fluctuations inside conducting walls and the image forces acting on the electrolyte particles, without any ad hoc separation Ansatz used in usual macroscopic treatments [25]. We used this strategy to show the screening effect of Casimir forces for semi-infinite geometries in paper [27], in what follows referred to as I.

The aim of the present work is to extend the study of the screening (cancellation) phenomena to universal Casimir terms appearing in the large-size expansions of the grand potential (1.2) for the Coulomb systems confined in fully-finite 2D geometries, in particular the disc geometry. The confining disc walls as well as the electrolyte inside the disc are modelled by two different microscopic two-component plasmas of point-like particles with $+/-$ unit charges in thermal equilibrium. Two cases are solved exactly: the high-temperature (Debye-Hückel) limit $\beta \to 0$ and the Thirring free-fermion point $\beta = 2$ corresponding to the collapse of positive and negative pairs of point-like charges. From the technical point of view, the circular symmetry of the problem leads to infinite summations over specific products of modified Bessel functions which have to be evaluated by using the asymptotic Debye expansion; we apply a few technicalities which help us to simplify a relatively complicated algebra. Similarities and differences with respect to screening phenomena in semi-infinite geometries, described in paper I, are pointed out.

The model is defined as follows. We shall consider Coulomb-gas systems of point-
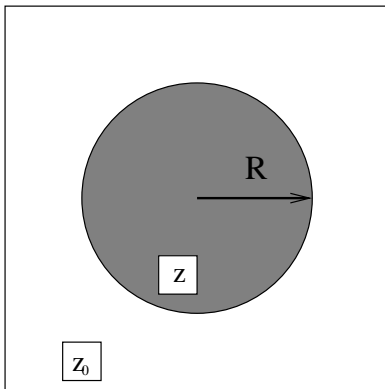
Figure 1. Disc geometry.

like particles with symmetric $+/-$ unit charges. Thermal equilibrium is treated in the grand canonical ensemble characterized by the inverse temperature $\beta$ and by the couple of equivalent (there is no external electrostatic potential), possibly position-dependent, particle fugacities $z_+(\mathbf{r}) = z_-(\mathbf{r}) = z(\mathbf{r})$. In the disc geometry presented in figure 1, there are two domains: the disc of radius $R$, $\Lambda_R = \{\mathbf{r}, |\mathbf{r}| < R\}$, and its complement $\bar{\Lambda}_R = \{\mathbf{r}, |\mathbf{r}| \geq R\}$. The electrolyte in $\Lambda_R$ is modelled by a two-component plasma with the particle fugacity denoted by $z$. The wall $\bar{\Lambda}_R$ is modelled by another two-component plasma with the particle fugacity $z_0$; the choice $z_0 > 0$ corresponds to a conducting wall ($\epsilon \to \infty$) while $z_0 = 0$ corresponds to vacuum (plain hard wall with $\epsilon = 1$). In 2D, the Coulomb potential $v$ at a spatial position $\mathbf{r}$, induced by a unit charge at the origin $\mathbf{0}$, is the solution of the 2D Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r}). \tag{1.3}$$

Explicitly,

$$v(\mathbf{r}) = -\ln\left(|\mathbf{r}|/a\right) \tag{1.4}$$

where $a$ is a free length scale, which only determines the zero of the potential and should not enter statistical mean values. The interaction energy of charged particles $\{i, q_i\}$, immersed in a homogeneous medium of dielectric constant $= 1$, is $\sum_{i<j} q_i q_j v(|\mathbf{r}_i - \mathbf{r}_j|)$.

The paper is organized as follows.

In general section 2, the Coulomb gas confined to some 2D regions is shown to be equivalent, in the ideal-conductor limit, to the massless Gaussian model defined in the complementary empty regions. Since in the derivation [14] of the large-size expansion of the free energy (1.1) for the critical Gaussian model with $c = 1$ only the curvature of the boundary is used, this explains the difference in sign of the logarithmic term between (1.1) and (1.2).

In section 3, the underlying 2D fully-finite Coulomb system is solved in the Debye-Hückel limit. All possible realizations of the model are considered, and the cancellation of the universal Casimir terms is documented when both the disc domain and its complement are occupied by a Coulomb gas. Fundamental differences between fully-finite and semi-infinite geometries are pointed out.

The exact solution of the model system at the Thirring free-fermion point $\beta = 2$ is presented in section 4. It is shown that basic features of Casimir phenomena, predicted by the mean-field theory, persist also at this specific temperature.

A brief recapitulation is given in section 5.

## 2. The Gaussian model approach

The universal term $(\chi/6) \ln R$ in the grand potential (1.2) of a finite 2D Coulomb system closely resembles the logarithmic term in (1.1) valid for critical systems. At first sight, it is surprising that a Coulomb system, with short-range particle correlations, exhibits a critical-like behaviour. It has been argued [16, 17] that this is related to the existence of long-ranged critical-like correlations of the electric potential; reasons for the difference in sign of the logarithmic term between (1.1) and (1.2) have been given. A slightly different, and we believe more convincing, argument will now be presented.

### 2.1. General formalism

In the present subsection, we consider a 2D manifold on which some regions $C$ are classical ideal conductors and some regions $E$ are empty. The ideal conductors are obtained as the infinite fugacity limit of conductors having some microscopic structure, which amounts to take the limit in which the microscopic length scale goes to zero. Thus, the results which will be obtained in this limit are expected to be valid when the length scales under consideration are much larger than the microscopic lengths. For simplicity, we restrict ourselves to an infinite-plane manifold, $C \cup E = R^2$, and start with conductors which are symmetric Coulomb gases of the kind described in the Introduction, but some generalizations are straightforward.

The interaction energy $E_N$ of $\pm 1$ charged particles $\{q_j, \mathbf{r}_j\}_{j=1}^N$ is expressible in terms of the microscopic charge density $\hat{\rho}_N(\mathbf{r}) = \sum_{j=1}^N q_j \delta(\mathbf{r} - \mathbf{r}_j)$ as follows

$$E_N(\{q_j, \mathbf{r}_j\}) = \frac{1}{2} \int_{R^2} \mathrm{d}^2 r \int_{R^2} \mathrm{d}^2 r' \hat{\rho}_N(\mathbf{r}) v(|\mathbf{r} - \mathbf{r}'|) \hat{\rho}_N(\mathbf{r}') - \frac{1}{2} N v(0), \quad (2.1)$$

where $v(0)$ is the (diverging) self-energy. The thermodynamic properties of the Coulomb system are determined by the grand partition function $\Xi$ defined as follows

$$\Xi = \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{z_+^{N_+} z_-^{N_-}}{N_+! \, N_-!} Q(N_+, N_-), \quad (2.2a)$$

where

$$Q(N_+, N_-) = \int_C \prod_{j=1}^N \mathrm{d}^2 r_j \exp\left[-\beta E_N(\{q_j, \mathbf{r}_j\})\right] \quad (2.2b)$$

is the configuration integral of $N_+$ positive and $N_-$ negative charges, $N = N_+ + N_-$ and $z_+ = z_- = z$ are the equivalent fugacities of charged particles. Let us insert the energy representation (2.1) into (2.2b). The self-energy simply renormalizes the particle fugacities. With regard to the fact that, according to (1.3), $-\Delta/(2\pi)$ is the

inverse operator of the Coulomb potential $v(\mathbf{r})$, the standard Hubbard-Stratonovich transformation (see e.g. ref. [28]) can be used to express the integral bilinear term as

$$\exp\left[-\frac{\beta}{2}\int_{R^2}\mathrm{d}^2r\int_{R^2}\mathrm{d}^2r'\hat{\rho}_N(\mathbf{r})v(|\mathbf{r}-\mathbf{r}'|)\hat{\rho}_N(\mathbf{r}')\right]$$
$$=\frac{\int\mathcal{D}\phi\exp\left[\int_{R^2}\mathrm{d}^2r\left(\frac{\beta}{4\pi}\phi\Delta\phi+\mathrm{i}\beta\phi\hat{\rho}_N\right)\right]}{\int\mathcal{D}\phi\exp\left(\int_{R^2}\mathrm{d}^2r\frac{\beta}{4\pi}\phi\Delta\phi\right)}, \tag{2.3}$$

where $\phi(\mathbf{r})$ is a real scalar field and $\int\mathcal{D}\phi$ denotes the functional integration over this field. The consequent factorization of the contributions from $(N_+, N_-)$ particle states in (2.2$a$) then allows to express the grand partition function of the system in the form [29]

$$\Xi=\frac{\int\mathcal{D}\phi\exp(-S[z(\mathbf{r})])}{\int\mathcal{D}\phi\exp(-S[0])}, \tag{2.4a}$$

with

$$S[z(\mathbf{r})]=\int_{C\cup E}\mathrm{d}^2r\left[\frac{\beta}{4\pi}\phi(\mathbf{r})(-\Delta)\phi(\mathbf{r})-2z(\mathbf{r})\cos(\beta\phi(\mathbf{r}))\right]. \tag{2.4b}$$

being the 2D Euclidean action of the sine-Gordon theory and

$$z(\mathbf{r})=\begin{cases} z & \text{for }\mathbf{r}\in C, \\ 0 & \text{for }\mathbf{r}\in E. \end{cases} \tag{2.5}$$

The normalization of $z$ is fixed by the short-distance expansion of the two-point correlation function

$$\langle\mathrm{e}^{\mathrm{i}\beta\phi(\mathbf{r})}\mathrm{e}^{-\mathrm{i}\beta\phi(\mathbf{r}')}\rangle\sim|\mathbf{r}-\mathbf{r}'|^{-\beta}\qquad\text{as }|\mathbf{r}-\mathbf{r}'|\to0 \tag{2.6}$$

under which the self-energy factor disappears from statistical relations; for more details see ref. [30] and the references cited therein.

Since the scalar $\phi$-field has to be regular at infinity, the term $\phi(-\Delta)\phi$ in (2.4$b$) can be transformed via the integration by parts into $|\nabla\phi|^2$. The sine-Gordon action (2.4$b$) thus takes its minimum at a $\phi(\mathbf{r})$ constant in space. Due to a discrete symmetry $\phi\to\phi+2\pi n/\beta$ with any integer $n$, the action has infinitely many ground states $|0_n\rangle$ characterized by the associate expectation values of the field $\langle\phi\rangle_n=2\pi n/\beta$. These ground states become all degenerate when the size of the Coulomb domain $|C|$ is large [31]. It is therefore sufficient to develop the sine-Gordon action (2.4$b$), on the classical level as well as on higher quantum-correction levels, around any one of these ground states [31], say $|0_0\rangle$ with $\langle\phi\rangle_0=0$. Now, in the ideal-conductor limit $z\to\infty$, the minimum of $-2z\cos(\beta\phi)$ around $\phi=0$ becomes infinitely sharp and $\phi(\mathbf{r})$ identically vanishes in the regions $C$. Up to an irrelevant multiplicative constant, the numerator of (2.4$a$) becomes

$$Z_{\mathrm{G}}=\int\mathcal{D}\phi\exp\left[-\frac{\beta}{4\pi}\int_E\mathrm{d}^2r\phi(\mathbf{r})(-\Delta)\phi(\mathbf{r})\right], \tag{2.7}$$

i.e. the partition function of the massless Gaussian model in the empty region(s). The boundary condition for $\phi$ is that it vanishes at the interface(s) of $E$ and $C$, since it

has to be continuous and vanishes in $C$. The denominator of $(2.4a)$ is just an (infinite) constant, independent of the geometry of the regions $E$ and $C$.

Although $\phi$ has some resemblance with the electric potential, it also has some drastically different properties. For instance, in a bulk Coulomb gas, the $\phi$ of the sine-Gordon representation has short-range two-point correlations while the electric potential has long-range ones [32]. Here, the electric potential has fluctuations in the conductor(s) (these fluctuations survive in the ideal-conductor limit) and by continuity the electric potential has fluctuations at the interface(s) of $E$ and $C$, while $\phi$ is identically zero at the interface(s).

### 2.2. Parallel conducting plates in $\nu$ dimensions

In the case of two parallel conducting plates, at a distance $d$ of each other, separated by vacuum, the Gaussian model (2.7), generalized to $\nu$ dimensions, has been shown [10, 17] to have in its free energy per unit area of one plate a universal $d$-dependent term $F$ such that the corresponding electrostatic Casimir force per unit area $-\partial F/\partial d$ is attractive and given by

$$-\beta\frac{\partial F}{\partial d} = -\frac{s_{\nu-1}\Gamma(\nu)\zeta(\nu)}{(2\pi)^{\nu-1}(2d)^\nu},\tag{2.8}$$

where $s_\nu$ is the area of the unit sphere in $\nu$ dimensions and where $\Gamma$ and $\zeta$ are the gamma function and the Riemann zeta function, respectively. In the Appendix, we give another derivation of (2.8) for the Gaussian model. In subsection 2.2. of I, we have obtained the same Casimir force (2.8) by using the fluctuations of the electric potential (which do not vanish at the surface of the plates). This is a check about the validity of the present method using the Gaussian model with simple Dirichlet boundary conditions for the $\phi$ field.

### 2.3. A hole in a 2D ideal conductor

We now come back to two dimensions. We consider an ideal conductor filling the whole plane except for an empty hole of characteristic size $R$, with a smooth boundary. It has been shown [14] that the Gaussian model has a free energy of the form (1.1) with $c = 1$ and $\chi = 1$. Thus, $\beta$ times the grand potential of the system has the universal term $-(1/6)\ln R$. If the hole is a disc in a plane, $R$ may be chosen as its radius.

### 2.4. A 2D Coulomb gas surrounded by an ideal conductor

We now consider the opposite geometry of an ideal conductor filling some connected region $C$ of the plane, with a smooth boundary, surrounded by vacuum. Now the relevant Gaussian model fills the exterior of $C$. In the derivation of ref. [14], the curvature of the boundary is used, and it is easy to see that the present geometry differs from the one of Section 2.3 by a change of sign of the curvature, leading to the universal term $+(1/6)\ln R$ in (1.2).

## 3. Debye-Hückel theory

### 3.1. General result

The high-temperature limit of the model system in figure 1 is described by the Debye-Hückel theory. For the general formalism, see e.g. section 3.1. of paper I. The formalism applied to the 2D symmetric Coulomb gas can be briefly summarized as follows.

In the bulk (homogeneous) regime, the total particle number density $n$ as a function of the fugacity $z$ is given by

$$\frac{n^{1-(\beta/4)}}{za^{\beta/2}} = 2 \left(\frac{\pi\beta}{2}\right)^{\beta/4} \exp\left(\frac{\beta C}{2}\right), \tag{3.1}$$

where $C$ is the Euler's constant. The inverse Debye length is defined by $\kappa^2 = 2\pi\beta n$.

In the inhomogeneous regime, the whole system domain $\Lambda$ can be separated into disjunct physically non-equivalent subdomains, $\Lambda = \cup_\alpha \Lambda^{(\alpha)}$. Within the grand canonical formalism, each subdomain is characterized by a constant fugacity, $z(\mathbf{r}) = z^{(\alpha)}$ for $\mathbf{r} \in \Lambda^{(\alpha)}$; the choice $z^{(\alpha)} = 0$ corresponds to a vacuum subdomain with no particles allowed to occupy the space. The corresponding "bulk" particle density $n^{(\alpha)}$ is related to the particle fugacity $z^{(\alpha)}$ via (3.1), and the corresponding inverse Debye length is $\kappa_\alpha = (2\pi\beta n^{(\alpha)})^{1/2}$. One introduces the screened Coulomb potential $G$ which obeys within each domain $\Lambda^{(\alpha)}$ the differential equation

$$\left[\Delta_1 - \kappa_\alpha^2\right] G(\mathbf{r}_1, \mathbf{r}_2) = -2\pi\delta(\mathbf{r}_1 - \mathbf{r}_2), \qquad \mathbf{r}_1 \in \Lambda^{(\alpha)}. \tag{3.2}$$

Here, the spatial position of the source point $\mathbf{r}_2$ is arbitrary. These equations are supplemented by the usual electrostatic conditions at each subdomain boundary $\partial\Lambda^{(\alpha)}$: $G$ and its normal derivative with respect to the boundary surface, $\partial_\perp G$, are continuous at $\partial\Lambda^{(\alpha)}$. The leading $\beta$-correction to the constant particle density $n^{(\alpha)}$ in the subdomain $\Lambda^{(\alpha)}$ is then determined by linearizing the exponential in the expression

$$n^{(\alpha)}(\mathbf{r}) = 2z^{(\alpha)} \exp\left\{\frac{\beta}{2} \lim_{\mathbf{r}' \to \mathbf{r}} \left[-G(\mathbf{r}, \mathbf{r}') + v(|\mathbf{r} - \mathbf{r}'|)\right]\right\} \tag{3.3}$$

where the 2D Coulomb potential $v$ is defined in equation (1.4).

We now apply the above formalism to the geometry of interest presented in figure 1. The inverse Debye length will be denoted by $\kappa$ for the disc domain $\Lambda_R = \{\mathbf{r}, r < R\}$ and by $\kappa_0$ for the complement wall domain $\bar{\Lambda}_R = \{\mathbf{r}, r > R\}$. We first consider the case when both $\kappa$ and $\kappa_0$ are nonzero. Let the source point $\mathbf{r}_2$ be first in the disc domain, i.e. $r_2 < R$. Equations (3.2) then take the form

$$\left[\Delta_1 - \kappa^2\right] G(\mathbf{r}_1, \mathbf{r}_2) = -2\pi\delta(\mathbf{r}_1 - \mathbf{r}_2), \qquad (r_1 < R), \tag{3.4a}$$

$$\left[\Delta_1 - \kappa_0^2\right] G(\mathbf{r}_1, \mathbf{r}_2) = 0, \qquad (r_1 > R). \tag{3.4b}$$

In terms of polar coordinates, the solution of these equations can be written as an expansion of the form

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{l=-\infty}^{\infty} \left[I_l(\kappa r_<)K_l(\kappa r_>) + a_l I_l(\kappa r_1)I_l(\kappa r_2)\right] \exp[il(\varphi_1 - \varphi_2)], \quad (r_1 < R), \tag{3.5a}$$

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{l=-\infty}^{\infty} b_l K_l(\kappa_0 r_1) I_l(\kappa r_2) \exp[il(\varphi_1 - \varphi_2)], \qquad (r_1 > R). \quad (3.5b)$$

Here, $r_< = \min\{r_1, r_2\}$ and $r_> = \max\{r_1, r_2\}$. $I_l$ and $K_l$ are modified Bessel functions possessing the symmetry $I_l = I_{-l}$, $K_l = K_{-l}$. They satisfy the same differential equation

$$f'' + \frac{1}{x}f' - \left(1 + \frac{l^2}{x^2}\right)f = 0, \qquad f = I_l(x) \text{ or } K_l(x), \quad (3.6)$$

but exhibit different asymptotic behaviours [33]:

$$I_l(x) \underset{x\to\infty}{\sim} \frac{e^x}{\sqrt{2\pi x}}, \qquad K_l(x) \underset{x\to\infty}{\sim} \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \quad (3.7)$$

and

$$I_l(x) \underset{x\to 0}{\sim} \frac{1}{|l|!}\left(\frac{x}{2}\right)^{|l|}, \qquad K_l(x) \underset{x\to 0}{\sim} \frac{|l|!}{2|l|}\left(\frac{x}{2}\right)^{-|l|}, \quad (3.8)$$

except for the special $l = 0$ case of $K_0(x) \sim -\ln(x/2) - C$ in the limit $x \to 0$ where $C$ is Euler's constant. We see that, in representations (3.5$a$) and (3.5$b$) of $G$, these asymptotic behaviours ensure the regularity of $G$ at the origin and at infinity. The coefficients $a_l$ and $b_l$ are determined by the above defined boundary conditions for $G(\mathbf{r}_1, \mathbf{r}_2)$ at $r_1 = R$:

$$K_l(\kappa R) + a_l I_l(\kappa R) = b_l K_l(\kappa_0 R), \quad (3.9a)$$

$$\kappa\left[K_l'(\kappa R) + a_l I_l'(\kappa R)\right] = \kappa_0 b_l K_l'(\kappa_0 R). \quad (3.9b)$$

Using the recursion formulas for the modified Bessel functions

$$xI_l'(x) = xI_{l\pm 1}(x) \pm lI_l(x), \quad (3.10a)$$

$$xK_l'(x) = -xK_{l\pm 1}(x) \pm lK_l(x), \quad (3.10b)$$

and the Wronskian relation [33]

$$I_l(x)K_{l+1}(x) + I_{l+1}(x)K_l(x) = \frac{1}{x}, \quad (3.11)$$

equations (3.9$a$) and (3.9$b$) give

$$a_l = -\frac{K_l(\kappa R)}{I_l(\kappa R)} + \frac{1}{RW_l}\frac{K_l(\kappa_0 R)}{I_l(\kappa R)}, \quad (3.12a)$$

$$b_l = \frac{1}{RW_l}, \quad (3.12b)$$

where the auxiliary quantity $W_l$ is given by

$$\begin{aligned}
W_l &= \kappa I_l'(\kappa R)K_l(\kappa_0 R) - I_l(\kappa R)\kappa_0 K_l'(\kappa_0 R) \\
&= \kappa I_{l+1}(\kappa R)K_l(\kappa_0 R) + I_l(\kappa R)\kappa_0 K_{l+1}(\kappa_0 R) \\
&= \kappa I_{l-1}(\kappa R)K_l(\kappa_0 R) + I_l(\kappa R)\kappa_0 K_{l-1}(\kappa_0 R).
\end{aligned} \quad (3.13)$$

Note the symmetries $a_l = a_{-l}$, $b_l = b_{-l}$ and $W_l = W_{-l}$. With regard to the differential equation (3.6) obeyed by the modified Bessel functions, it is easy to show that $W_l$ fulfills the equality

$$\frac{\partial}{\partial R}\ln(RW_l) = \frac{1}{W_l}(\kappa^2 - \kappa_0^2)I_l(\kappa R)K_l(\kappa_0 R). \quad (3.14)$$

The same procedure can be applied when the source point $\mathbf{r}_2$ lies outside of the disc, i.e. $r_2 > R$. For $r_1 > R$, one obtains the solution of the form

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{l=-\infty}^{\infty} \left[ I_l(\kappa_0 r_<) K_l(\kappa_0 r_>) + c_l K_l(\kappa_0 r_1) K_l(\kappa_0 r_2) \right] \exp[il(\varphi_1 - \varphi_2)]. \qquad (3.15)$$

The coefficients $c_l$ are determined by the boundary conditions for $G$ at $r_1 = R$ as follows

$$c_l = -\frac{I_l(\kappa_0 R)}{K_l(\kappa_0 R)} + \frac{1}{RW_l}\frac{I_l(\kappa R)}{K_l(\kappa_0 R)}. \qquad (3.16)$$

They fulfill the symmetry $c_l = c_{-l}$.

To obtain the density profile, note that in relations (3.5$a$) and (3.15) the first terms in the sums over $l$ correspond to the expansion of the modified Bessel functions $K_0(\kappa|\mathbf{r}_1 - \mathbf{r}_2|)$ and $K_0(\kappa_0|\mathbf{r}_1 - \mathbf{r}_2|)$, respectively. In formula (3.3) for the particle density, these bulk contributions, minus the pure Coulomb potential $v$, imply the density-fugacity relationship (3.1). The second terms in the sums over $l$ in equations (3.5$a$) and (3.15) are "reflected" contributions due to the boundary at $r = R$. After the linearization of (3.3) in $\beta$, they lead to

$$n(r) = n - \frac{\beta n}{2}\sum_{l=0}^{\infty}\mu_l a_l I_l^2(\kappa r), \qquad (r < R), \qquad (3.17a)$$

$$n(r) = n_0 - \frac{\beta n_0}{2}\sum_{l=0}^{\infty}\mu_l c_l K_l^2(\kappa_0 r), \qquad (r > R). \qquad (3.17b)$$

Here, $\mu_l$ is the Neumann factor: $\mu_0 = 1$ and $\mu_l = 2$ for $l \geq 1$.

Having at one's disposal the density profile, the grand canonical partition function $\Xi_R(\kappa, \kappa_0)$ can be deduced in the following way. For the present geometry, $\Xi_R$ is defined by

$$\Xi_R = \sum_{N_+, N_-=0}^{\infty} \frac{1}{N_+! N_-!} \prod_{i=1}^{N}\left(\int_0^R d^2 r_i\, z + \int_R^{\infty} d^2 r_i\, z_0\right)\exp\left[-\beta\sum_{i<j}q_i q_j v(|\mathbf{r}_i - \mathbf{r}_j|)\right], \quad (3.18)$$

where $N_+$ ($N_-$) is the number of positively (negatively) charged particles and $N = N_+ + N_-$. The averaged particle density at position $\mathbf{r}$ is given by $n(\mathbf{r}) = \langle\sum_i \delta(\mathbf{r} - \mathbf{r}_i)\rangle$. Combining this with equation (3.18), one finds that

$$\frac{\partial}{\partial R}\ln\Xi_R = 2\pi R\left[n(R^-) - n(R^+)\right], \qquad (3.19)$$

where $R^-$ ($R^+$) means the left (right) limit to $R$. With respect to the density profile [equations (3.17$a$) and (3.17$b$) with coefficients $a_l$ and $c_l$ given by (3.12$a$) and (3.16), respectively], this relation takes the explicit form

$$\frac{\partial}{\partial R}\ln\Xi_R = 2\pi R(n - n_0) - \frac{1}{2}(\kappa^2 - \kappa_0^2)\sum_{l=0}^{\infty}\mu_l\frac{1}{W_l}I_l(\kappa R)K_l(\kappa_0 R)$$

$$+ \frac{R}{2}\sum_{l=0}^{\infty}\mu_l\left[\kappa^2 I_l(\kappa R)K_l(\kappa R) - \kappa_0^2 I_l(\kappa_0 R)K_l(\kappa_0 R)\right]. \qquad (3.20)$$

The integration with respect to $R$ of the second term on the rhs of (3.20) can be done by using (3.14) and the integration of the last term follows from the indefinite integration

formula for Bessel functions [34]

$$\int \mathrm{d}x \, x I_l(x) K_l(x) = \frac{x^2}{4} \left[ 2 I_l(x) K_l(x) + I_{l+1}(x) K_{l-1}(x) + I_{l-1}(x) K_{l+1}(x) \right]$$
$$= \frac{1}{2} \left[ (x^2 + l^2) I_l(x) K_l(x) - x^2 I_l'(x) K_l'(x) \right]. \tag{3.21}$$

Thence, from (3.20) one gets

$$\ln \Xi_R(n, n_0) = \text{const} + \pi R^2 (n - n_0) - \frac{1}{2} \sum_{l=0}^{\infty} \mu_l \ln (RW_l)$$

$$+ \frac{1}{4} \sum_{l=0}^{\infty} \mu_l \left\{ \left[ (\kappa R)^2 + l^2 \right] I_l(\kappa R) K_l(\kappa R) - (\kappa R)^2 I_l'(\kappa R) K_l'(\kappa R) \right\} \tag{3.22}$$

$$- \frac{1}{4} \sum_{l=0}^{\infty} \mu_l \left\{ \left[ (\kappa_0 R)^2 + l^2 \right] I_l(\kappa_0 R) K_l(\kappa_0 R) - (\kappa_0 R)^2 I_l'(\kappa_0 R) K_l'(\kappa_0 R) \right\}.$$

The integration constant is fixed by considering the $R \to 0$ limit. With the aid of the asymptotic formulas (3.8) one gets, for instance, $\lim_{R \to 0}(RW_l) = (\kappa/\kappa_0)^{|l|}$. After simple algebra, the final result reads

$$\ln \Xi_R(n, n_0) = \ln \Xi_{R=0} + \pi R^2 (n - n_0) - \frac{1}{2} \sum_{l=0}^{\infty} \mu_l \ln \left[ RW_l \left( \frac{\kappa_0}{\kappa} \right)^l \right]$$

$$+ \frac{1}{4} \sum_{l=0}^{\infty} \mu_l \left\{ \left[ (\kappa R)^2 + l^2 \right] I_l(\kappa R) K_l(\kappa R) - (\kappa R)^2 I_l'(\kappa R) K_l'(\kappa R) - l \right\} \tag{3.23}$$

$$- \frac{1}{4} \sum_{l=0}^{\infty} \mu_l \left\{ \left[ (\kappa_0 R)^2 + l^2 \right] I_l(\kappa_0 R) K_l(\kappa_0 R) - (\kappa_0 R)^2 I_l'(\kappa_0 R) K_l'(\kappa_0 R) - l \right\}.$$

Here, $\Xi_{R=0}$ is the grand canonical partition function of the system with zero disc radius, given by the obvious relation $\lim_{\Lambda \to \infty} \ln \Xi_{R=0}/|\Lambda| = \beta p(n_0) = n_0[1 - (\beta/4)]$. Formula (3.23) was derived under the assumption that both particle densities $n$ and $n_0$ are nonzero.

The limit $n \to 0$ corresponds to no particles present inside the disc, i.e. the vacuum disc hole surrounded by the fluctuating conductor wall. Taking the $n \to 0$ limit in (3.23), one gets

$$\ln \Xi_R(n = 0, n_0) = \ln \Xi_{R=0} - \pi R^2 n_0 - \frac{1}{2} \sum_{l=0}^{\infty} \mu_l \ln \left[ 2 \left( \frac{\kappa_0 R}{2} \right)^{l+1} \frac{1}{l!} K_{l+1}(\kappa_0 R) \right] \tag{3.24}$$

$$- \frac{1}{4} \sum_{l=0}^{\infty} \mu_l \left\{ \left[ (\kappa_0 R)^2 + l^2 \right] I_l(\kappa_0 R) K_l(\kappa_0 R) - (\kappa_0 R)^2 I_l'(\kappa_0 R) K_l'(\kappa_0 R) - l \right\}.$$

The limit $n_0 \to 0$ corresponds to no particles present outside of the disc, i.e. the Coulomb system in the disc surrounded by vacuum (the plain hard wall). One can take the $n_0 \to 0$ limit in every term of equation (3.23), except for the $l = 0$ term in the first summation on the rhs of that equation. This $l = 0$ term makes a problem due to the logarithmic divergence of $K_0(\kappa_0 R)$ in $W_0$ in the limit $\kappa_0 \to 0$. The problem with the $l = 0$ term, caused by the fact that the effective Coulomb potential $G(r)$ is not screened at asymptotically large distances $r$ in the present limit $\kappa_0 \to 0$, was discussed e.g. in [35]). According to this reference, when calculating the particle density (3.17a) one finds

that the coefficient $a_0$ depends on the free length scale $a$ of the 2D Coulomb potential (1.4). Since a statistical mean value should not depend on $a$, the limit $a \to \infty$, which puts the zero of the Coulomb potential to infinity, has to be considered. Our formalism in the limit $\kappa_0 \to 0$ is equivalent to that outlined in [35] if one sets $\kappa_0 = 1/a$. Thus, returning to the particle density (3.17$a$), the auxiliary quantity $W_l$ (3.13), considered in the limit $\kappa_0 \to 0$ and for $R > 0$, behaves like

$$W_l \sim \kappa I_{l-1}(\kappa R) K_l(\kappa_0 R) \qquad \text{for } l \geq 0, \tag{3.25}$$

and the coefficients $a_l$, given by (3.12$a$), take the form

$$a_l = \frac{K_{l-1}(\kappa R)}{I_{l-1}(\kappa R)}, \qquad l = 0, 1, \ldots. \tag{3.26}$$

On the basis of (3.19) taken with $n(R^+) \equiv 0$, it holds

$$\frac{\partial}{\partial R} \ln \Xi_R(n, n_0 = 0) = 2\pi R n - \frac{\kappa^2 R}{2} \sum_{l=0}^{\infty} \mu_l \frac{K_{l-1}(\kappa R)}{I_{l-1}(\kappa R)} I_l^2(\kappa R). \tag{3.27}$$

The integration of this equation finally implies

$$\ln \Xi_R(n, n_0 = 0) = \pi R^2 n - \frac{1}{2} \ln \left[ (\kappa R) I_1(\kappa R) \right] + \text{const}$$
$$- \sum_{l=1}^{\infty} \ln \left[ \left( \frac{2}{\kappa R} \right)^{l-1} (l-1)! I_{l-1}(\kappa R) \right] \tag{3.28}$$
$$+ \frac{1}{4} \sum_{l=0}^{\infty} \mu_l \left\{ \left[ (\kappa R)^2 + l^2 \right] I_l(\kappa R) K_l(\kappa R) - (\kappa R)^2 I_l'(\kappa R) K_l'(\kappa R) - l \right\}.$$

Here, since the grand partition function depends on the length scale $a = 1/\kappa_0$, the integration constant is in fact infinite in the considered limit $a \to \infty$. It is clear from the derivation procedure that formula (3.28) is valid only for $R > 0$, and it cannot serve as a basis for an expansion around the $R = 0$ point.

### 3.2. Large-R analysis

Each of the above derived grand potentials $\Omega = -(1/\beta) \ln \Xi$ is given in terms of infinite sums which cannot be summed up explicitly. What can be done is the evaluation of first few terms of the asymptotic expansion of the sums for large disc radius $R \to \infty$. Denoting by $\alpha$ either of the dimensionless combinations $\kappa R$ or $\kappa_0 R$, one has to use the Debye expansion [36] of the modified Bessel functions $I_l(\alpha), K_l(\alpha)$ and of their derivatives, since this expansion is valid for large $l$ *uniformly* with respect to $\alpha/l$. In particular, one has

$$I_l(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{(\alpha^2 + l^2)^{1/4}} e^{\eta} \left[ 1 + \frac{3t - 5t^3}{24l} + O\left( \frac{1}{\alpha^2 + l^2} \right) \right], \tag{3.29a}$$

$$K_l(\alpha) = \sqrt{\frac{\pi}{2}} \frac{1}{(\alpha^2 + l^2)^{1/4}} e^{-\eta} \left[ 1 - \frac{3t - 5t^3}{24l} + O\left( \frac{1}{\alpha^2 + l^2} \right) \right], \tag{3.29b}$$

and

$$I_l'(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{(\alpha^2 + l^2)^{1/4}}{\alpha} e^\eta \left[ 1 + \frac{-9t + 7t^3}{24l} + O\left(\frac{1}{\alpha^2 + l^2}\right) \right], \qquad (3.30a)$$

$$K_l'(\alpha) = -\sqrt{\frac{\pi}{2}} \frac{(\alpha^2 + l^2)^{1/4}}{\alpha} e^{-\eta} \left[ 1 - \frac{-9t + 7t^3}{24l} + O\left(\frac{1}{\alpha^2 + l^2}\right) \right], \quad (3.30b)$$

where

$$\eta(l, \alpha) = \sqrt{\alpha^2 + l^2} - l \sinh^{-1}(l/\alpha), \qquad t = l/\sqrt{\alpha^2 + l^2}.$$

The consequent sums over $l$ can be performed by applying the (generalized) Euler-MacLaurin summation formula [36]:

$$\sum_{l=m}^{n} f(l) = \int_m^n f(l) \mathrm{d}l + \frac{1}{2}[f(n) + f(m)] + \frac{B_2}{2!}[f'(n) - f'(m)] + \frac{B_4}{4!}[f'''(n) - f'''(m)] + \cdots, (3.31)$$

where $B_\nu$ are Bernoulli numbers: $B_2 = 1/6$, $B_4 = -1/30$, etc.

In the case of the Coulomb gas of particle density $n_0$, localized outside of the disc of radius $R$ with vacuum in the disc hole, formula (3.24) evaluated in the $R \to \infty$ limit implies

$$\beta\Omega_R(n = 0, n_0) = -\beta p(n_0)(|\Lambda| - \pi R^2) + \beta\gamma(n_0) \, 2\pi R - \frac{1}{6}\ln(\kappa_0 R) + O(1). \qquad (3.32)$$

The first term on the rhs of (3.32) is the bulk contribution with the pressure $p$ given by

$$\beta p(n_0) = \left(1 - \frac{\beta}{4}\right) n_0, \qquad (3.33)$$

the second term is the surface contribution with the surface tension $\gamma$ given, in the Debye-Hückel limit, by [27]

$$\beta\gamma(n_0) = \int_0^\infty \frac{\mathrm{d}l}{4\pi} \ln\left[ \frac{(l + \sqrt{\kappa_0^2 + l^2})^2}{4l\sqrt{\kappa_0^2 + l^2}} \right] = \frac{\kappa_0}{8\pi}(4 - \pi). \qquad (3.34)$$

Finally, the third logarithmic term has the universal coefficient $-1/6$. This Casimir term, caused by electrostatic fluctuations inside the wall, tends to *dilate* the empty disc domain. This is the fundamental difference in comparison with semi-infinite geometries where fluctuating walls *attract* one another.

In the case of the Coulomb gas of particle density $n$, localized inside the disc of radius $R$ and surrounded by vacuum, formula (3.28) evaluated in the $R \to \infty$ limit gives

$$\beta\Omega_R(n, n_0 = 0) = -\beta p(n) \pi R^2 + \beta\gamma(n) \, 2\pi R + \frac{1}{6}\ln(\kappa R) + O(1). \qquad (3.35)$$

As before, the first and second terms on the rhs of (3.35) correspond to the volume and surface parts of $\beta\Omega$, respectively. The third logarithmic term has the coefficient $1/6$, in agreement with the general relation (1.2) taken at the disc value of $\chi = 1$. This coefficient is opposite to the one in (3.32) which confirms the relation of the universal term to the curvature of the constraining surface. It is important to note that in semi-infinite geometries (see e.g. paper I) the grand potential of a Coulomb system

constrained by vacuum plain hard walls does not exhibit the universal Casimir term. This is another fundamental difference between fully-finite and semi-infinite geometries.

From a technical point of view, it is useful to sum the two expressions for the grand potential (3.24) and (3.28), taken at the same particle density denoted say by $\bar{n}$:

$$\beta\Omega_R(n = \bar{n}, n_0 = 0) + \beta\Omega_R(n = 0, n_0 = \bar{n}) = \beta\Omega_{R=0} + \text{const}$$
$$+\frac{1}{2}\ln\left[(\bar{\kappa}R)^2 K_1(\bar{\kappa}R)I_1(\bar{\kappa}R)\right] + \sum_{l=1}^{\infty}\ln\left[\frac{1}{2l}(\bar{\kappa}R)^2 I_{l-1}(\bar{\kappa}R)K_{l+1}(\bar{\kappa}R)\right]. \quad (3.36)$$

Using first the recursion formulas (3.10) and subsequently the asymptotic expansions (3.29) and (3.30) for the modified Bessel functions, the application of the Euler-MacLaurin summation formula (3.31) implies after simple algebra that

$$\beta\Omega_R(n = \bar{n}, n_0 = 0) + \beta\Omega_R(n = 0, n_0 = \bar{n}) = \beta\Omega_{R=0} + 2\beta\gamma(\bar{n})2\pi R + O(1). \quad (3.37)$$

This relation proves the consistency of asymptotic expansions (3.32) and (3.35), and enables us to write down one knowing the explicit form of the other. Simplifying technicalities of this kind will be used in what follows.

We are now ready for studying the large-$R$ asymptotic behaviour of the grand potential when both particle densities $n$ and $n_0$ are nonzero, see formula (3.23). One has for the specific combination of grand potentials

$$\beta\Omega_R(n, n_0) - [\beta\Omega_R(n, n_0 = 0) + \beta\Omega_R(n = 0, n_0)] = \text{const} + \frac{1}{2}\ln\left[\frac{W_0}{\kappa\kappa_0 R I_1(\kappa R)K_1(\kappa_0 R)}\right]$$
$$-\sum_{l=1}^{\infty}\ln\left\{\frac{[\kappa R I_{l-1}(\kappa R)][\kappa_0 R K_{l+1}(\kappa_0 R)]}{2lRW_l}\right\}. \quad (3.38)$$

As above, using the recursion formulas and the asymptotic expansions for the modified Bessel functions, the Euler-MacLaurin summation formula leads to

$$\beta\Omega_R(n, n_0) - [\beta\Omega_R(n, n_0 = 0) + \beta\Omega_R(n = 0, n_0)]$$
$$= R\int_0^{\infty} dl \ln\left[\frac{2l\left(\sqrt{\kappa^2 + l^2} + \sqrt{\kappa_0^2 + l^2}\right)}{\left(l + \sqrt{\kappa^2 + l^2}\right)\left(l + \sqrt{\kappa_0^2 + l^2}\right)}\right] + O(1). \quad (3.39)$$

With regard to the asymptotic expansions (3.32) and (3.35), one finally arrives at

$$\beta\Omega_R(n, n_0) = -\beta p(n)\pi R^2 - \beta p(n_0)(|\Lambda| - \pi R^2) + \beta\gamma(n, n_0)2\pi R + O(1), \quad (3.40)$$

where

$$\beta\gamma(n, n_0) = \int_0^{\infty}\frac{dl}{4\pi}\ln\left[\frac{\left(\sqrt{\kappa^2 + l^2} + \sqrt{\kappa_0^2 + l^2}\right)^2}{4\sqrt{\kappa^2 + l^2}\sqrt{\kappa_0^2 + l^2}}\right] \quad (3.41)$$

is the (dimensionless) contact surface tension of the two 2D plasmas in the Debye-Hückel limit, see equation (3.26) of paper I. The universal logarithmic Casimir term does not appear in (3.40), it is "screened". This phenomenon is intuitively expected: since there are Coulomb systems on both sides of the disc boundary, the curvature contributions with opposite signs cancel with one another. The same cancellation of long-ranged Casimir forces takes place in semi-infinite geometries, see paper I.

## 4. The free-fermion point

*4.1. General result*

The 2D Coulomb gas of symmetric unit charges is exactly solvable at the collapse point $\beta = 2$; for general formalism see e.g. section 4.1. of paper I. In the grand canonical formalism, at $\beta = 2$, the bulk system is characterized by the rescaled particle fugacity $m = 2\pi az$ [$a$ is a free length scale introduced in (1.2)] which has the dimension of an inverse length. The many-particle densities can be expressed in terms of specific Green functions $G_{qq'}(\mathbf{r}, \mathbf{r}')$ ($q, q' = \pm$); because of the symmetry between positive and negative particles one only needs $G_{++}$ and $G_{-+}$. These Green functions are determined by the equations

$$(\Delta_1 - m^2)G_{++}(\mathbf{r}_1, \mathbf{r}_2) = -m\delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{4.1}$$

and

$$G_{-+}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{m}\left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial y_1}\right)G_{++}(\mathbf{r}_1, \mathbf{r}_2), \tag{4.2}$$

supplemented by the vanishing boundary conditions when $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$. In infinite space, the solution of (4.1) reads

$$G_{++}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi}K_0(m|\mathbf{r}_1 - \mathbf{r}_2|). \tag{4.3}$$

The one-particle densities $n_+ = n_- = n/2$ ($n$ is the total particle density), given by

$$n_q(\mathbf{r}) = mG_{qq}(\mathbf{r}, \mathbf{r}), \tag{4.4}$$

are infinite since $K_0(mr)$ diverges logarithmically as $r \to 0$. Regularization of the Coulomb interaction by a short-distance cutoff $L$ implies for the particle density

$$n = \frac{m^2}{\pi}K_0(mL) \underset{mL \to 0}{\sim} \frac{m^2}{\pi}\left[\ln\left(\frac{2}{mL}\right) - C\right]. \tag{4.5}$$

In the inhomogeneous case when the system domain $\Lambda = \cup_\alpha \Lambda^{(\alpha)}$, each subdomain $\Lambda^{(\alpha)}$ is characterized by a constant rescaled fugacity, $m(\mathbf{r}) = m_\alpha$ for $\mathbf{r} \in \Lambda^{(\alpha)}$, and the corresponding bulk density $n_\alpha$ defined as a function of $m_\alpha$ by (4.5). Within each domain, the Green function $G_{++}$ obeys the differential equation

$$(\Delta_1 - m_\alpha^2)G_{++}(\mathbf{r}_1, \mathbf{r}_2) = -m_\alpha\delta(\mathbf{r}_1 - \mathbf{r}_2), \qquad \mathbf{r}_1 \in \Lambda^{(\alpha)}, \tag{4.6}$$

where the spatial position of the source point $\mathbf{r}_2$ is arbitrary. $G_{-+}$ is determined by relation (4.2) with $m$ substituted by the subdomain-dependent $m(\mathbf{r}_1)$. The boundary conditions are that both $G_{++}$ and $G_{-+}$ must be continuous at each subdomain boundary $\partial\Lambda^{(\alpha)}$. The one-particle densities are again given by (4.4) with $m \to m(\mathbf{r})$.

For the disc geometry of figure 1, the rescaled particle fugacity is equal to $m$ in the disc domain $r < R$ and to $m_0$ in the complementary wall $r > R$. Let the source point $\mathbf{r}_2$ be first in the disc domain, i.e. $r_2 < R$. Equations (4.1) then take the form

$$(\Delta_1 - m^2)G_{++}(\mathbf{r}_1, \mathbf{r}_2) = -m\delta(\mathbf{r}_1 - \mathbf{r}_2), \qquad (r_1 < R), \tag{4.7a}$$
$$(\Delta_1 - m_0^2)G_{++}(\mathbf{r}_1, \mathbf{r}_2) = 0, \qquad (r_1 > R). \tag{4.7b}$$

In terms of polar coordinates, the solution of these equations is written as an expansion of the form

$$G_{++}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} \sum_{l=-\infty}^{\infty} [I_l(mr_<)K_l(mr_>)$$

$$+ a_l I_l(mr_1)I_l(mr_2)] \exp[il(\varphi_1 - \varphi_2)], \qquad (r_1 < R), \qquad (4.8a)$$

$$G_{++}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} \sum_{l=-\infty}^{\infty} b_l K_l(m_0 r_1)I_l(mr_2) \exp[il(\varphi_1 - \varphi_2)], \qquad (r_1 > R). \qquad (4.8b)$$

The polar version of relation (4.2) for $G_{-+}$ takes the form

$$G_{-+}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{e^{i\varphi_1}}{m(\mathbf{r}_1)}\left(\frac{\partial}{\partial r_1} + \frac{i}{r_1}\frac{\partial}{\partial \varphi_1}\right)G_{++}(\mathbf{r}_1, \mathbf{r}_2). \qquad (4.9)$$

The coefficients $a_l$ and $b_l$ are determined by the continuity conditions for $G_{++}(\mathbf{r}_1, \mathbf{r}_2)$ and $G_{-+}(\mathbf{r}_1, \mathbf{r}_2)$ at the disc boundary $r_1 = R$:

$$K_l(mR) + a_l I_l(mR) = b_l K_l(m_0 R), \qquad (4.10a)$$

$$K_{l+1}(mR) - a_l I_{l+1}(mR) = b_l K_{l+1}(m_0 R). \qquad (4.10b)$$

The solution reads

$$a_l = -\frac{K_l(mR)}{I_l(mR)} + \frac{1}{mRV_l}\frac{K_l(m_0 R)}{I_l(mR)}, \qquad (4.11a)$$

$$b_l = \frac{1}{mRV_l}, \qquad (4.11b)$$

where

$$V_l = I_l(mR)K_{l+1}(m_0 R) + I_{l+1}(mR)K_l(m_0 R). \qquad (4.12)$$

Note the symmetry $V_l = V_{-l-1}$. It can be readily shown that the auxiliary quantity $V_l$ fulfills the equation

$$\frac{\partial}{\partial R}\ln(RV_l) = \frac{m - m_0}{V_l}[I_l(mR)K_l(m_0 R) + I_{l+1}(mR)K_{l+1}(m_0 R)]. \qquad (4.13)$$

One proceeds analogously when the source point $\mathbf{r}_2$ lies in the wall, i.e. $r_2 > R$. For the case $r_1 > R$, one gets

$$G_{++}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m_0}{2\pi} \sum_{l=-\infty}^{\infty} [I_l(m_0 r_<)K_l(m_0 r_>)$$

$$+ c_l K_l(m_0 r_1)K_l(m_0 r_2)] \exp[il(\varphi_1 - \varphi_2)], \qquad (r_1, r_2 > R). \qquad (4.14)$$

The coefficients $c_l$ are given by the boundary conditions at $r_1 = R$ as follows

$$c_l = -\frac{I_l(m_0 R)}{K_l(m_0 R)} + \frac{1}{m_0 RV_l}\frac{I_l(mR)}{K_l(m_0 R)}. \qquad (4.15)$$

The density at $r \to R^-$, determined by (4.4), (4.8a) and (4.11a), reads

$$n(R^-) = \frac{m}{\pi R} \sum_{l=-\infty}^{\infty} \frac{1}{V_l}I_l(mR)K_l(m_0 R). \qquad (4.16)$$

The sum in (4.16) is divergent, and so it has to be formally regularized by taking an upper cutoff on $l$. Similarly, the density at $r \to R^+$, determined by (4.4), (4.14) and (4.15), is expressible as follows

$$n(R^+) = \frac{m_0}{m} n(R^-). \tag{4.17}$$

The grand canonical partition function $\Xi_R(m, m_0)$ is again determined by the differential equation (3.19), in particular

$$\frac{\partial}{\partial R} \ln \Xi_R = 2(m - m_0) \sum_{l=0}^{\infty} \frac{1}{V_l} \left[ I_l(mR) K_l(m_0 R) + I_{l+1}(mR) K_{l+1}(m_0 R) \right]. \tag{4.18}$$

Due to the equality (4.13), this equation can be integrated to the form

$$\ln \Xi_R(m, m_0) = \text{const} + 2 \sum_{l=0}^{\infty} \ln(RV_l). \tag{4.19}$$

The integration constant is fixed by the $R \to 0$ limit. Since $\lim_{R \to 0} RV_l = m_0^{-1}(m/m_0)^l$ for $l \geq 0$, one gets

$$\ln \Xi_R(m, m_0) = \ln \Xi_{R=0} + 2 \sum_{l=0}^{\infty} \ln \Bigg\{ \left( \frac{m_0}{m} \right)^l m_0 R$$

$$\times \left[ I_l(mR) K_{l+1}(m_0 R) + I_{l+1}(mR) K_l(m_0 R) \right] \Bigg\}. \tag{4.20}$$

Here, $\Xi_{R=0}$ is the grand canonical partition function of the system with zero disc radius, i.e. $\lim_{\Lambda \to \infty} \ln \Xi_{R=0}/|\Lambda| = \beta p(n_0)$.

Although (4.20) was derived under the assumption that both rescaled fugacities $m$ and $m_0$ are nonzero, there is no problem to consider the zero limit of either of fugacities directly in (4.20). One obtains

$$\ln \Xi_R(m, m_0 = 0) = 2 \sum_{l=0}^{\infty} \ln \left[ l! \left( \frac{2}{mR} \right)^l I_l(mR) \right] \tag{4.21}$$

and

$$\ln \Xi_R(m = 0, m_0) = \ln \Xi_{R=0} + 2 \sum_{l=0}^{\infty} \ln \left[ \frac{2}{l!} \left( \frac{m_0 R}{2} \right)^{l+1} K_{l+1}(m_0 R) \right]. \tag{4.22}$$

### 4.2. Large-R analysis

The result (4.21) for the 2D two-component plasma at $\beta = 2$, in the disc surrounded by vacuum, has already been derived in [16]. There, the large-$R$ asymptotic form of the grand potential was derived in the form

$$\beta \Omega_R(m, m_0 = 0) = -\beta p(m) \pi R^2 + \beta \gamma(m) 2\pi R + \frac{1}{6} \ln(mR) + O(1), \tag{4.23}$$

where $p(m)$ is the regularized pressure and the surface tension $\gamma(m)$ is given by

$$\beta \gamma(m) = \int_0^{\infty} \frac{\mathrm{d}l}{2\pi} \ln \left( \frac{2\sqrt{m^2 + l^2}}{l + \sqrt{m^2 + l^2}} \right) = m \left( \frac{1}{4} - \frac{1}{2\pi} \right). \tag{4.24}$$

One can apply the standard procedure to formula (4.22) corresponding to the plasma outside of the empty disc, to obtain the large-$R$ asymptotic behaviour

$$\beta\Omega_R(m=0, m_0) = -\beta p(m_0)(|\Lambda| - \pi R^2) + \beta\gamma(m_0)2\pi R - \frac{1}{6}\ln(m_0 R) + O(1). \qquad (4.25)$$

One sees that, similarly as in the Debye-Hückel limit, in comparison with (4.23) the universal logarithmic term has the opposite sign.

Technically, it is simpler to derive (4.25) by summing up the two basic expressions (4.21) and (4.22) taken at the same rescaled fugacity, say $\bar{m}$:

$$\beta\Omega_R(m=\bar{m}, m_0=0) + \beta\Omega_R(m=0, m_0=\bar{m})$$
$$= \beta\Omega_{R=0} - 2\sum_{l=0}^{\infty} \ln\left[\bar{m}R I_l(\bar{m}R) K_{l+1}(\bar{m}R)\right]. \qquad (4.26)$$

The standard asymptotic procedure for the modified Bessel functions leads to the following asymptotic behaviour

$$\beta\Omega_R(m=\bar{m}, m_0=0) + \beta\Omega_R(m=0, m_0=\bar{m}) = \beta\Omega_{R=0} + 2\beta\gamma(\bar{m})2\pi R + O(1), \qquad (4.27)$$

which proves the consistency of relations (4.23) and (4.25).

Finally, considering the special combination of grand potentials

$$\beta\Omega_R(m, m_0) - [\beta\Omega_R(m, m_0=0) + \beta\Omega_R(m=0, m_0)]$$
$$= -2\sum_{l=0}^{\infty} \ln\left[1 + \frac{I_{l+1}(mR)K_l(m_0 R)}{I_l(mR)K_{l+1}(m_0 R)}\right], \qquad (4.28)$$

and using the standard asymptotic procedure, one gets

$$\beta\Omega_R(m, m_0) - [\beta\Omega_R(m, m_0=0) + \beta\Omega_R(m=0, m_0)]$$
$$= R\int_0^{\infty} dl \ln\left[\frac{\left(l + \sqrt{m^2 + l^2}\right)\left(l + \sqrt{m_0^2 + l^2}\right)}{2\left(mm_0 + \sqrt{m^2 + l^2}\sqrt{m_0^2 + l^2} + l^2\right)}\right] + O(1). \qquad (4.29)$$

With regard to the asymptotic relations (4.23) and (4.25), one concludes that

$$\beta\Omega_R(m, m_0) = -\beta p(m)\pi R^2 - \beta p(m_0)(|\Lambda| - \pi R^2) + \beta\gamma(m, m_0)2\pi R + O(1), \qquad (4.30)$$

where

$$\beta\gamma(m, m_0) = \int_0^{\infty} \frac{dl}{2\pi}\ln\left(\frac{2\sqrt{m^2 + l^2}\sqrt{m_0^2 + l^2}}{mm_0 + \sqrt{m^2 + l^2}\sqrt{m_0^2 + l^2} + l^2}\right) \qquad (4.31)$$

defines the surface tension of the two 2D plasmas at $\beta = 2$, see equation (4.12) of paper I. As was expected, the universal logarithmic term disappears once again.

We have verified that the basic features of the mean-field behaviour, predicted by the Debye-Hückel analysis in the previous section, persist also at the specific inverse temperature $\beta = 2$.

## 5. Conclusion

The name "ideal-conductor" usually means that the electric potential has some constant value (for instance zero) inside the conductor, without any fluctuations. More precisely, we call such a model of a conductor "inert ideal-conductor". The present paper deals with microscopic models of conductors, in which there are fluctuations of the electric potential. They are "living" conductors. In the limit when the microscopic lengths, such as the Debye length, go to zero (in practice, are small compared to the macroscopic length, here such as the radius $R$), the fluctuations of the electric potential survive. We call a conductor in that high-density limit "living ideal-conductor".

In a previous publication [17], Coulomb systems with inert ideal-conductor boundary conditions were studied‡. In this model, for a Coulomb system in a slab of width $d$, a repulsive Casimir force was found, which is just the opposite of (2.8) which holds for living ideal-conductor plates separated by vacuum. For a Coulomb system in a slab with living ideal-conductor walls, the two contributions cancel each other, leaving only a short-range attraction, as shown in I.

Similarly, in ref. [17], a 2D Coulomb system in a disc of radius $R$ with inert ideal-conductor walls was found to have a logarithmic universal contribution $(1/6) \ln R$ to $\beta$ times its grand potential. This is just the opposite of the $-(1/6) \ln R$ found here for an empty circular hole surrounded by a Coulomb gas. Again, for a Coulomb system in a disc with living conductor walls, the two contributions cancel each other.

## Acknowledgments

## Appendix

In ref. [17], (2.8) has been derived by using the quantum Hamiltonian in $\nu - 1$ dimensions. Here, we give an alternative, more direct, derivation.

The free energy per unit area $F$ corresponding to the Gaussian partition function (2.7), generalized to $\nu$ dimensions, is given, up to an irrelevant additive constant, by

$$\beta F = -\frac{1}{L^{\nu-1}} \ln Z_{\mathrm{G}} = \frac{1}{2L^{\nu-1}} \sum \ln \lambda, \tag{A.1}$$

where $L$ is the linear size of a plate and $\lambda$ the eigenvalues of minus the Laplacian. The planes, on which $\phi$ obeys Dirichlet boundary conditions, are perpendicular to the $x$ axis at $x = 0$ and $x = d$. Each point is defined by its Cartesian coordinates $(x, \mathbf{r}^\perp)$ where $\mathbf{r}^\perp$ is a $(\nu - 1)$-dimensional vector normal to the $x$ axis. In this geometry, in the limit of $L$ infinite, the eigenfunctions of $-\Delta$ are $\exp(-i\mathbf{l} \cdot \mathbf{r}^\perp) \sin(n\pi x/d)$ where $\mathbf{l}$ is a wavevector

‡ In [17], the Coulomb potential in $\nu$ dimensions ($\nu > 2$) was defined as $r^{2-\nu}$ while in I it was defined as $r^{2-\nu}/(\nu - 2)$. These different definitions do not change the Casimir terms under consideration.

normal to the $x$ axis and $n$ an integer larger than 0. The corresponding eigenvalues are $\lambda = l^2 + (n\pi/d)^2$. Therefore, (A.1) can be written as

$$\beta F = \frac{1}{2} \int \frac{\mathrm{d}^{\nu-1}l}{(2\pi)^{\nu-1}} \sum_{n=1}^{\infty} \ln\left(l^2 + \left(\frac{n\pi}{d}\right)^2\right). \tag{A.2}$$

Thus

$$-\beta\frac{\partial F}{\partial d} = \int \frac{\mathrm{d}^{\nu-1}l}{(2\pi)^{\nu-1}} \frac{1}{d} \sum_{n=1}^{\infty} \frac{(n\pi/d)^2}{l^2 + (n\pi/d)^2}. \tag{A.3}$$

The sum on $n$ in (A.3) diverges. The divergent part can be separated by noting that the summand minus 1 is $-(ld/\pi)^2/[(ld/\pi)^2 + n^2]$, which can be explicitly summed [33]. The result for the total sum, including the divergent part, is

$$\frac{1}{d}\sum_{n=1}^{\infty} \frac{(n\pi/d)^2}{l^2 + (n\pi/d)^2} = -\frac{1}{2}l\coth(ld) + \frac{1}{2d} + \sum_{n=1}^{\infty}\frac{1}{d}. \tag{A.4}$$

The last two terms of (A.4) can be regrouped into the divergent sum $S = (1/2)\sum_{n=-\infty}^{\infty}(1/d)$. Since $F$ has a bulk term of the form $Ad$, with $A$ infinite in the present ideal conductor limit, for obtaining the finite-$d$ Casimir force $f$, we must subtract from (A.3) its value for $d$ infinite. For $d$ infinite, the term $n\pi/d$ in the eigenvalue $\lambda$ becomes the continuous wave number $k$ and the divergent sum $S$ becomes $\int_{-\infty}^{\infty} \mathrm{d}k/(2\pi)$. Since, for $d$ finite, the summand in $S$ is a constant, $S$ can also be written as the same integral on $k$, and after the subtraction the divergent term disappears. Finally,

$$\beta f = -\left[\beta\frac{\partial F}{\partial d} - \beta\frac{\partial F}{\partial d}\bigg|_{d=\infty}\right]$$
$$= -\frac{1}{2}\int \frac{\mathrm{d}^{\nu-1}l}{(2\pi)^{\nu-1}} l[\coth(ld) - 1] = -\frac{s_{\nu-1}\Gamma(\nu)\zeta(\nu)}{(2\pi)^{\nu-1}(2d)^\nu}, \tag{A.5}$$

in agreement with (2.8).

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